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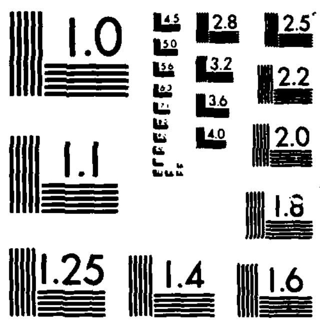

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A CONVERGENCE THEOREM FOR NEWTON'S METHOD IN BANACH SPACES

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ABSTRACT

On the basis of the results obtained in a series of papers [25] - [28], a convergence theorem for Newton's method in Banach spaces is given, which improves the theorems of Kantorovich [4], Lancaster [8] and Ostrowski [16]. The error bounds obtained improve the recent results of Potra [17].

AMS(MOS) Subject Classifications: 65G99, 65J15

Key Words: convergence theorem, Newton's method, Kantorovich's theorem, Lancaster's theorem, Ostrowski's theorem, error estimates, Potra's bounds

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SIGNIFICANCE AND EXPLANATION

To find sharper error bounds for iterative solution of nonlinear equations under assumptions as weak as possible is of basic importance in numerical analysis. This paper gives a convergence theorem for Newton's method in Banach spaces which improves the theorems of Kantorovich [4], Lancaster [8] and Ostrowski [16]. The error bounds obtained improve the recent results of Potra [17].

A CONVERGENCE THEOREM FOR NEWTON'S METHOD IN BANACH SPACES

Tetsuro Yamamoto*

1. Introduction

There is much literature concerning convergence and error estimates for Newton's method in Banach spaces. In a series of papers [25] - [28], we examined the error bounds which have been obtained by many authors (Dennis [1], Tapia [24], Rall-Tapia [20], Ostrowski [15], [16], Gragg-Tapia [3], Miel [9] - [11], Potra-Pták [18], Moret [12]) un the assumptions of the Kantorovich theorem, and compared them with the Kantorovich bounds. As the result, we concluded [28] that their results follow from the Kantorov theorem so that, under the Kantorovich assumptions, the Kantorovich theorem still give the best upper bounds for the Newton method.

In this paper, we are interested in improving the assumptions of the Kantorovich theorem and the assertions of the Ostrowski theorem [16; Theorem 38.1]. We shall first state both theorems and several lemmas in §2. Next, in §3, we shall present a convergence theorem which improves both theorems. It will also be shown that results improve the error bounds of Lancaster [8], Kornstaedt [7] and Potra [17]. Finally, in §4, we shall show that Ostrowski's other theorem [16; Theorem 38.2] can be derived by our approach.

2. Preliminaries

Let X and Y be Banach spaces and consider an operator $F : D \subseteq X \to Y$. If F is Fréchet differentiable in an open convex set $D_0 \subseteq D$, then the Newton method for solving the equation

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$$F(x) = 0 \tag{2.1}$$

is defined by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) , \quad n \geq 0 , \tag{2.2}$$

provided that $F'(x_n)^{-1} \in L(Y,X)$ exists at each step, where $L(Y,X)$ denotes the Banach space of bounded linear operators of Y into X. Sufficient conditions for convergence of the iterates (2.2), error estimates and existence and uniqueness regions of solutions are given by the famous Kantorovich theorem:

Theorem 2.1 (Kantorovich [4], [5] and Kantorovich-Akilov [6]). Let $F : D \subseteq X \to Y$ be Frécht differentiable in an open convex set $D_0 \subseteq D$. Assume that for some $x_0 \in D_0$, $F'(x_0)$ is invertible and that

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq K\|x - y\| , \quad K > 0, \quad x, y \in D_0 ,$$

$$\|F'(x_0)^{-1}F(x_0)\| \leq \eta , \quad \eta > 0 ,$$

$$h = K\eta \leq \frac{1}{2}$$

and

$$\bar{S}(x_0,t^*) = \{x \in X | \; \|x - x_0\| \leq t^* = \frac{1 - \sqrt{1 - 2h}}{K}\} \subseteq D_0 .$$

Then:

(i) The iterates (2.2) are well-defined, lie in the open ball $S(x_0,t^*) = \{x \in X | \|x - x_0\| < t^*\}$ and converge to a solution x^* of the equation (2.1).

(ii) The solution x^* is unique in $S(x_0,t^{**}) \cap D_0$ if $2h < 1$ and in $\bar{S}(x_0,t^{**})$ if $2h = 1$, where $t^{**} = (1 + \sqrt{1 - 2h})/K$.

(iii) Error estimates

$$\|x^* - x_n\| \leq \frac{2\eta_n}{1 + \sqrt{1 - 2h_n}} \leq 2^{1-n}(2h)^{2^n-1}\eta , \quad n \geq 0, \tag{2.3}$$

hold, where η_n and h_n are defined by the recurrence relations

$$B_0 = 1, \quad \eta_0 = \eta, \quad h_0 = h = K\eta,$$

$$B_n = \frac{B_{n-1}}{1 - h_{n-1}}, \quad \eta_n = \frac{h_{n-1}\eta_{n-1}}{2(1 - h_{n-1})}, \quad h_n = KB_n\eta_n, \quad n \geq 1. \tag{2.4}$$

(iii)' Put $f(t) = \frac{1}{2} Kt^2 - t + \eta$ and define the sequence $\{t_n\}$ by

$$t_0 = 0, \quad t_{n+1} = t_n - f(t_n)/f'(t_n), \quad n \geq 0.$$

Then

$$\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$$

and the error estimate

$$\|x^* - x_n\| \leq t^* - t_n, \quad n \geq 0, \tag{2.5}$$

holds.

The bounds (2.3) are of the form found in [4], while the bound (2.5) is found in [5] and [6]. We should remark here that B_n and η_n are the bounds for $\|F'(x_n)^{-1}F'(x_0)\|$ and $\|x_{n+1} - x_n\|$, respectively. In fact, by induction on n, we have

$$\|F'(x_n)^{-1}F'(x_0)\| = \|\{I + F'(x_{n-1})^{-1}(F'(x_n) - F'(x_{n-1}))\}^{-1}F'(x_{n-1})^{-1}F'(x_0)\|$$

$$\leq \frac{B_{n-1}}{1 - B_{n-1}K\|x_n - x_{n-1}\|} \leq \frac{B_{n-1}}{1 - h_{n-1}} = B_n,$$

$$\begin{aligned} x_{n+1} - x_n &= -F'(x_n)^{-1}F(x_n) \\ &= -F'(x_n)^{-1}\{F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1})\} \\ &= -F'(x_n)^{-1}\int_0^1 \{F'(x_{n-1} + t(x_n - x_{n-1})) - F'(x_{n-1})\}(x_n - x_{n-1})dt \end{aligned} \tag{2.6}$$

and

$$\|x_{n+1} - x_n\| \leq \frac{1}{2} B_n K\|x_n - x_{n-1}\|^2 \leq \frac{1}{2} B_n K\eta_{n-1}^2 = \eta_n.$$

On the other hand, Ostrowski [15], [16] proved the convergence of the Newton method under the assumptions which are slightly different from those of Kantorovich.

Theorem 2.2 (Ostrowski [15; Theorems 38.1 and 40.2]). Let $F: D \subseteq X \to Y$ and D^o be the interior of D. Assume that for some $x_0 \in D^o$, $F'(x_0)$ and $F'(x_0)^{-1}$ exist. Let $\varphi \geq 0$, $\alpha = 1 + \cosh \varphi$, $\rho = e^{-\varphi}\|x_1 - x_0\|$ and $\sigma = \alpha \|F(x_0)\| \cdot \|F'(x_0)^{-1}\|^2$. Consider the line segment $L = \{tx_0 + (1-t)x_1 \,|\, 0 \leq t \leq 1\}$ and the closed ball $\bar{S} = \bar{S}(x_1, \rho)$, and put $C = L \cup \bar{S}$. Assume now that $C \subseteq D^o$, F is Fréchet differentiable on C and

$$\|F'(x) - F'(y)\| \leq \frac{1}{\sigma}\|x - y\|, \quad x, y \in L, \quad x, y \in \bar{S}. \tag{2.7}$$

Then the Newton iterates (2.2) are well-defined, $x_n \in \bar{S}$, $n \geq 1$, and $\{x_n\}$ converges to a solution $x^* \in \bar{S}$ of (2.1), which is unique in C. Furthermore, the following inequalities hold:

$$\begin{aligned}\|x^* - x_n\| &\leq e^{-2^{n-1}\varphi} \frac{\sinh \varphi}{\sinh 2^{n-1}\varphi}\|x_1 - x_0\| \\ &\leq 2^{1-n}\|x_1 - x_0\|,\end{aligned} \tag{2.8}$$

$$\|x^* - x_{n+1}\| \leq e^{-2^n\varphi}\|x_{n+1} - x_n\|, \quad n \geq 0. \tag{2.9}$$

In [28], we derived (2.8) and (2.9) under the assumptions of Theorem 2.1 and showed that they do not improve Gragg-Tapia's bounds. Furthermore, we proved that Moret's bounds, which also follow from Theorem 2.1, are sharper than those of Gragg-Tapia, Potra-Pták and Miel. The argument in [28] also works under the assumptions of an affine invariant version of Theorem 2.2, which are weaker than those of Theorem 2.1 with $\eta = \|x_1 - x_0\|$, provided that $x_0 \neq x_1$. Therefore, on the basis of results obtained in [28], we can improve Theorems 2.1 and 2.2.

Before giving an improved version of both theorems, we state several lemmas. In the following, without loss of generality, we assume that $F(x_0) \neq 0$. This assumption will be kept throughout this paper.

Lemma 2.1. Let $F: D \subseteq X \to Y$ and D^o be the interior of D. Assume that for some $x_0 \in D^o$, $F'(x_0)$ and $F'(x_0)^{-1}$ exist and $F(x_0) \neq 0$. Let $\varphi \geq 0$, $\alpha = 1 + \cosh \varphi$,

$\eta = \|x_1 - x_0\|$ and $\rho = e^{-\varphi}\eta$. Define the sets L, $\bar{S}$ and C as in Theorem 2.2. Furthermore, assume that $C \subseteq D^{\circ}$, F is Fréchet differentiable on C and, for some $K > 0$,

$$\|F'(x_0)^{-1}(F'(x) - F'(y))\| \leq K\|x - y\|, \quad x, y \in L, \ x, y \in \bar{S} \ , \tag{2.10}$$

and

$$\alpha h \equiv \alpha K\eta \leq 1 \ .$$

Then the iterates (2.2) are well-defined, $x_n \in S = S(x_1, \rho)$ (open ball), $n \geq 1$ and $\|x_{n+1} - x_n\| \leq t_{n+1} - t_n$, $n \geq 0$, where $\{t_n\}$ is the majorizing sequence defined in Theorem 2.1. Therefore, the sequence $\{x_n\}$ converges to a solution $x^* \in \bar{S}$ of (2.1) and $\|x^* - x_n\| \leq t^* - t_n$.

Proof. By the assumption $\alpha h \leq 1$, we have $2h \leq 1$ since $\alpha \geq 2$. Therefore, the majorant theory of Kantorovich can be applied to the sequence $\{x_n\}$, by noting that the condition (2.10) holds and that

$$\begin{aligned} \|x_{n+1} - x_1\| &\leq \|x_{n+1} - x_n\| + \|x_n - x_1\| \\ &\leq (t_{n+1} - t_n) + (t_n - t_1) = t_{n+1} - t_1 \\ &< t^* - t_1 = \frac{1 - h - \sqrt{1 - 2h}}{K} \\ &\leq e^{-\varphi}\eta = \rho \ , \end{aligned} \tag{2.11}$$

where equality holds in (2.11) if and only if $\alpha h = 1$. Q.E.D.

Lemma 2.2. Under the assumptions of Lemma 2.1, define the sequences $\{t_n\}$, $\{B_n\}$, $\{\eta_n\}$ and $\{h_n\}$ as in Theorem 2.1. Then

$$t_{n+1} - t_n = \eta_n \ ,$$

$$t^* - t_n = \frac{1 - \sqrt{1 - 2h_n}}{KB_n} = \frac{2\eta_n}{1 + \sqrt{1 - 2h_n}}$$

and

$$t^{**} - t_n = \frac{1 + \sqrt{1 - 2h_n}}{KB_n}, \quad n \geq 0 .$$

That is, $t^* - t_n$ and $t^{**} - t_n$ are the solutions of the equation

$\frac{1}{2} KB_n t^2 - t + \eta_n = 0.$

Proof. The same proof as in [28] works under the assumptions of Lemma 2.1. Q.E.D.

Lemma 2.3. *Under the assumptions of Lemma 2.2, we have for* $n \geq 1$

(i) $B_n^{-1} = 1 - Kt_n = \sqrt{1 - 2h + (K\eta_{n-1})^2}$

(ii) $\frac{1}{2} KB_n = \frac{\nabla t_{n+1}}{(\nabla t_n)^2}$,

(iii) $\frac{KB_n}{1 + \sqrt{1 - 2h_n}} = \frac{t^* - t_n}{(\nabla t_n)^2}$,

where ∇ denotes the backward difference operator.

Proof. See the proof of Proposition A.3 in [28]. Q.E.D.

Lemma 2.4. Under the assumptions of Lemma 2.2, let $\theta = t^*/t^{**} = (1 - \sqrt{1 - 2h})/(1 + \sqrt{1 - 2h})$. Then we have for $n \geq 0$

$$\text{(i)}\quad t^* - t_n = \begin{cases} \frac{2}{K} \sqrt{1 - 2h}\, \frac{\theta^{2^n}}{1 - \theta^{2^n}} & (2h < 1) \\ 2^{1-n}\eta & (2h = 1) , \end{cases}$$

$$\text{(ii)}\quad \frac{t^* - t_{n+1}}{\nabla t_{n+1}} = \theta^{2^n} .$$

Proof. See the proofs of Proposition A.1 and Proposition A.4 (ii) in [28]. Q.E.D.

Lemma 2.5. Under the notation and assumptions of Lemma 2.4, we have for $n \geq 0$

$$\text{(i)}\quad t^* - t_n \leq \begin{cases} e^{-2^{n-1}\varphi} \dfrac{\sinh \varphi}{\sinh 2^{n-1}\varphi} \eta & (\varphi > 0) \\ 2^{1-n}\eta = \lim\limits_{\varphi \to +0} \left(e^{-2^{n-1}\varphi} \dfrac{\sinh \varphi}{\sinh 2^{n-1}\varphi} \eta\right) & (\varphi = 0), \end{cases}$$

$$\text{(ii)}\quad \theta^{2^n} \leq e^{-2^n\varphi} .$$

The equalities hold in (i) and (ii) if and only if $\alpha h = 1$.

Proof. Take $\varphi^* \geq 0$ such that $\alpha^* h \equiv (1 + \cosh \varphi^*)h = 1$. Then, by Proposition A.4 in [28], we have that the equalities hold in (i) and (ii). Therefore Lemma 2.5 follows for every $\varphi \in [0,\varphi^*]$, since the right hand-sides of (i) and (ii) are monotone decreasing with respect to φ. Q.E.D.

We end this section by proving the following lemma.

Lemma 2.6. Under the assumptions of Lemma 2.1, define the sequence $\{B_n\}$ as in Theorem 2.1. If, for some n, there exists a constant $M_n > 0$ such that

$$\|x^* - x_{n+1}\| \leq \frac{1}{2} M_n \|x^* - x_n\|^2$$

and $M_n \leq KB_n$. Then

$$\|x^* - x_n\| \leq \sigma_n^* \equiv \frac{2d_n}{1 + \sqrt{1 - 2M_n d_n}},$$

where $d_n = \|x_{n+1} - x_n\|$.

Proof. Without loss of generality, we may assume that $d_n \neq 0$, that is $x^* \neq x_n$. Then, by assumptions, we have

$$\|x^* - x_n\| - d_n \leq \frac{1}{2} M_n \|x^* - x_n\|^2 \leq \frac{1}{2} KB_n \|x^* - x_n\|^2 .$$

Hence, if we put

$$\phi_n(t) = \frac{1}{2} M_n t^2 - t + d_n \quad \text{and} \quad \tilde{\phi}_n(t) = \frac{1}{2} KB_n t^2 - t + d_n ,$$

then

$$\tilde{\Phi}_n(\|x^* - x_n\|) \geq \Phi_n(\|x^* - x_n\|) \geq 0$$

and

$$\tilde{\Phi}_n(t) > \Phi_n(t) \quad \text{for} \quad t > 0 \quad \text{or} \quad \tilde{\Phi}_n(t) \equiv \Phi_n(t) \ .$$

By Lemmas 2.1 and 2.2, we have

$$\|x^* - x_n\| \leq t^* - t_n$$

and $t^* - t_n$, $t^{**} - t_n$ are two solutions of the equation $\Psi_n(t) = \frac{1}{2} KB_n t^2 - t + \eta_n = 0.$

Furthermore we have $\Psi_n(t) \geq \tilde{\Phi}_n(t)$. Therefore $\Phi_n(t)$ and $\tilde{\Phi}_n(t)$ have positive solutions σ_n^*, σ_n^{**} and $\tilde{\sigma}_n^*$, $\tilde{\sigma}_n^{**}$ respectively such that

$$\sigma_n^* < \tilde{\sigma}_n^* \leq t^* - t_n \leq t^{**} - t_n \leq \tilde{\sigma}_n^{**} < \sigma_n^{**} \quad \text{if} \quad M_n < KB_n \ ,$$

$$\sigma_n^* = \tilde{\sigma}_n^* < t^* - t_n \leq t^{**} - t_n < \tilde{\sigma}_n^{**} = \sigma_n^{**} \quad \text{if} \quad M_n = KB_n \quad \text{and} \quad d_n < \eta_n$$

and

$$\sigma_n^* = \tilde{\sigma}_n^* = t^* - t_n \leq t^{**} - t_n = \tilde{\sigma}_n^{**} = \sigma_n^{**} \quad \text{if} \quad M_n = KB_n \quad \text{and} \quad d_n = \eta_n \ .$$

In any case we have

$$\|x^* - x_n\| \leq \sigma_n^* \ ,$$

since $\Phi_n(\|x^* - x_n\|) \geq 0$ implies $\|x^* - x_n\| \leq \sigma_n^*$ or $\|x^* - x_n\| \geq \sigma_n^{**}$ and the latter case can be excluded. Q.E.D.

3. Results

We are now in a position to prove the following theorem.

Theorem 3.1. Under the assumptions of Lemma 2.1, the following results hold:

(i) The iterates (2.2) are well-defined, the sequence $\{x_n\}$, $n \geq 1$ remains in an open ball $S = S(x_1, \rho)$ and converges to a solution $x^* \in \bar{S}$ of the equation (2.1).

(ii) The solution is unique in C.

(iii) Let

$$K_n = \sup_{\substack{x,y\in \bar{S} \\ x\neq y}} \frac{\|F'(x_n)^{-1}(F'(x) - F'(y))\|}{\|x - y\|}, \quad n \geq 1,$$

$$K_0 = K$$

and put $d_n = \|x_{n+1} - x_n\|$. Then the following error estimates hold.

(a) A posteriori error estimates:

$$\|x^* - x_n\| \leq \frac{2d_n}{1 + \sqrt{1 - 2K_n d_n}} \quad (n \geq 0) \tag{3.1}$$

$$\leq \begin{cases} t^* = \dfrac{2n}{1 + \sqrt{1 - 2h}} & (n = 0) \\ \dfrac{2d_n}{1 + \sqrt{1 - 2K\{1 - K(\|x_n - x_1\| + d_0)\}^{-1} d_n}} & (n \geq 1) \end{cases} \tag{3.2}$$

$$\leq \frac{2d_n}{1 + \sqrt{1 - 2K(1 - Kt_n)^{-1} d_n}} \quad (n \geq 0) \tag{3.3}$$

$$\leq \frac{t^* - t_n}{\nabla t_{n+1}} d_n \quad (n \geq 0)$$

$$\leq \frac{t^* - t_n}{(\nabla t_n)^2} d_{n-1}^2 \quad (n \geq 1) \quad \text{(Miel [10])}$$

$$\leq \frac{Kd_{n-1}^2}{\sqrt{1 - 2h + (Kd_{n-1})^2} + \sqrt{1 - 2h}} \quad (n \geq 1) \quad \text{(Potra-Pták [18])}$$

$$\leq \frac{t^* - t_n}{\nabla t_n} d_{n-1} \quad (n \geq 1) \quad \text{(Miel [10])}$$

$$= \theta^{2^{n-1}} d_{n-1} \quad (n \geq 1) \quad \text{(Gragg-Tapia [3])}$$

$$\leq e^{-2^{n-1}\varphi} d_{n-1} \quad (n \geq 1) \quad \text{(Ostrowski [16])}. \tag{3.4}$$

The equality holds in (3.4) if and only if $\alpha h = 1$.

(b) A priori error estimates:

$$\|x^* - x_n\| \leq t^* - t_n \quad (n \geq 0) \quad \text{(Kantorovich [5], [6])}$$

$$= \begin{cases} \frac{2}{K}\sqrt{1-2h}\,\dfrac{\theta^{2^n}}{1-\theta^{2^n}} & (2h < 1) \\ 2^{1-n}\eta \quad (2h = 1) & (n \geq 0) \quad \text{(Gragg-Tapia [3])} \end{cases}$$

$$\leq \begin{cases} e^{-2^{n-1}\varphi}\,\dfrac{\sinh\varphi}{\sinh 2^{n-1}\varphi}\,\eta & (\varphi > 0) \\ 2^{1-n}\eta \quad (\varphi = 0) & (n \geq 0)\,. \quad \text{(Ostrowski [16])} \end{cases} \tag{3.5}$$

The equality holds in (3.5) if and only if $\alpha h = 1$.

(iv) If F is Fréchet differentiable in an open convex set D_0 such that $D^o \supseteq D_0 \supset C$ and if $F'(x)$ satisfies the Lipschitz condition in D_0 with the Lipschitz constant K, then the solution x^* is unique in

$$\tilde{S} = \begin{cases} S(x_0, t^{**}) \cap D_0 & \text{if } 2h < 1 \\ \bar{S}(x_0, t^{**}) \cap D_0 & \text{if } 2h = 1\,. \end{cases}$$

Furthermore, (3.2) may be replaced by the sharper bound

$$\|x^* - x_n\| \leq \frac{2d_n}{1 + \sqrt{1 - 2K(1 - K\Delta_n)^{-1} d_n}} \quad (n \geq 0) \quad \text{(Moret [12])} \tag{3.6}$$

where $\Delta_n = \|x_n - x_0\|$.

Proof. (i) was proved in Lemma 2.1. To prove (ii), let $\tilde{x}^*$ be a solution in $\bar{S}$. Then we have

$$\tilde{x}^* - x_{n+1} = \tilde{x}^* - x_n + F'(x_n)^{-1}F(x_n)$$

$$= - F'(x_n)^{-1}\{F(\tilde{x}^*) - F(x_n) - F'(x_n)(\tilde{x}^* - x_n)\}$$

$$= - F'(x_n)^{-1}F'(x_0) \int_0^1 F'(x_0)^{-1}\{F'(x_n + t(\tilde{x}^* - x_n)) - F'(x_n)\}(\tilde{x}^* - x_n)dt$$

and

$$x_n + t(\tilde{x}^* - x_n), \quad x_n \in \bar{S}, \quad n \geq 1.$$

Hence, by (2.10) and Lemma 2.3 (ii) we have

$$\|\tilde{x}^* - x_{n+1}\| \leq \frac{1}{2} B_n K\|\tilde{x}^* - x_n\|^2 = \frac{\nabla t_{n+1}}{(\nabla t_n)^2} \|\tilde{x}^* - x_n\|^2$$

so that

$$\frac{\|\tilde{x}^* - x_{n+1}\|}{\nabla t_{n+1}} \leq \left(\frac{\|\tilde{x}^* - x_n\|}{\nabla t_n}\right)^2 \leq \cdots \leq \left(\frac{\|\tilde{x}^* - x_1\|}{\nabla t_1}\right)^{2^n} \leq \left(\frac{\rho}{\eta}\right)^{2^n} \leq 1 .$$

This implies

$$\|\tilde{x}^* - x_{n+1}\| \leq \nabla t_{n+1} \to 0$$

as $n \to \infty$. Thus we obtain $\tilde{x}^* = \lim_{n\to\infty} x_n = x^*$. Next, we shall show that there is no solution in $C\backslash\bar{S}$, provided that the set $C\backslash\bar{S}$ is not empty. To show this, let $\bar{x}^*$ be a solution in $C\backslash\bar{S}$. Then we have

$$e^{-\varphi}\eta = \rho < \|\bar{x}^* - x_1\| \leq \frac{1}{2} K\|\bar{x}^* - x_0\|^2 < \frac{1}{2} K(\eta - \rho)^2 = \frac{1}{2} K(1 - e^{-\varphi})^2\eta^2$$

so that

$$1 < K (\cosh \varphi - 1)\eta = K(\alpha - 2)\eta \leq 1 - 2h ,$$

which is a contradiction. This proves the uniqueness of the solution in C. To prove (iii), we first observe that

$$x^* - x_{n+1} = -F'(x_n)\int_0^1 \{F'(x_n + t(x^* - x_n)) - F'(x_n)\}\,(x^* - x_n)dt$$

$$x_n + t(x^* - x_n),\quad x_n \in \bar{S},\quad n \geq 1 ,$$

$$F'(x_n)^{-1} = \{I + F'(x_0)^{-1}(F'(x_n) - F'(x_1)) + F'(x_0)^{-1}(F'(x_1) - F'(x_0))\}^{-1}F'(x_0)^{-1}$$

and

$$x_n,\ x_1 \in \bar{S},\quad n \geq 1,\quad x_1,\ x_0 \in L .$$

Hence we have from Lemma 2.3 (i)

$$\|F'(x_n)^{-1}F'(x_0)\| \leq B_n' \equiv \frac{1}{1 - K(\|x_n - x_1\| + d_0)} \quad (n \geq 1) \tag{3.7}$$

$$\leq \frac{1}{1 - Kt_n} = B_n$$

and

$$\|x^* - x_{n+1}\| \leq \frac{1}{2} K_n \|x^* - x_n\|^2 \leq \frac{1}{2} KB_n' \|x^* - x_n\|^2 \leq \frac{1}{2} KB_n \|x^* - x_n\|^2 .$$

Therefore Lemma 2.6 can be applied to obtain the bounds (3.1) - (3.3) for $n \geq 1$. Observe that if $n = 0$, they reduce to the Kantorovich bound $t^* = 2\eta/(1 + \sqrt{1 - 2h})$. The other part of (iii) follows from Lemmas 2.2 - 2.5. This proves (iii). Finally we shall prove (iv). Let F be Fréchet differentiable in an open convex set D_0 such that $D^o \supseteq D_0 \supset C$ and $F'(x)$ satisfy the Lipschitz condition (2.10) in D_0. Then (3.7) may be replaced by the sharper estimate

$$\|F'(x_n)^{-1}F'(x_0)\| \leq \frac{1}{1 - K\|x_n - x_0\|}, \quad n \geq 0 .$$

Therefore, Lemma 2.6 can again be applied to replace (3.2) by (3.6). To prove the uniqueness of solution in $\tilde{S}$, let $\tilde{x}^*$ be a solution in $\tilde{S}$. Then there exists a nonnegative constant r such that $r < 1$ if $2h < 1$, $r \leq 1$ if $2h = 1$ and $\|\tilde{x}^* - x_0\| \leq rt^{**}$. By induction on n, we can show that

$$\|\tilde{x}^* - x_n\| \leq r^{2^n}(t^{**} - t_n), \quad n \geq 0 . \tag{3.8}$$

In fact, we have under our assumptions

$$\|\tilde{x}^* - x_{n+1}\| \le \frac{1}{2} KB_n \|x^* - x_n\|^2$$
$$\le \frac{1}{2} KB_n \{r^{2^n}(t^{**} - t_n)\}^2$$
$$= r^{2^{n+1}}(t^{**} - t_n - \eta_n)$$
$$= r^{2^{n+1}}(t^{**} - t_{n+1}) ,$$

where we have used the induction hypothesis and Lemma 2.2. This proves (3.8), from which we obtain

$$\|\tilde{x}^* - x_n\| \to 0$$

as $n \to \infty$, since $r^{2^n} \to 0$ if $2h < 1$ and $t^{**} - t_n = t^* - t_n \to 0$ if $2h = 1$. Hence we have $\tilde{x}^* = \lim_{n\to\infty} x_n = x^*$, which implies the uniqueness of solution in $\tilde{S}$. Q.E.D.

Assumptions similar to those of Theorem 2.2 or Lemma 2.1 were also adopted by Lancaster [8], and later by Schmidt [22], [23] for the generalized secant method which includes the Newton method as a special case. Lancaster's assumptions corresond to the case $\varphi = 0$ in Lemma 2.1, while Schmidt's correspond to the case where φ is chosen so that $\alpha h = 1$, in which case we have $\rho = e^{-\varphi}\eta = t^* - \eta$. In the following we shall improve Lancaster's result. (In the case of the Newton method, Schmidt's upper bound reduces to the Kantorovich bound $t^* - t_n$. Also see [28].)

Corollary 3.1.1. Let $F: D \subseteq X \to Y$ and D^o be the interior of D. Assume that for some $x_0 \in D^o$, $F'(x_0)$ and $F'(x_0)^{-1}$ exist, $F'(x_0) \neq 0$ and F is Fréchet differentiable on $\bar{S}_0 = \bar{S}(x_1, \|x_1 - x_0\|)$. Furthermore, put

$$L_0 = \sup_{\substack{x,y\in\bar{S}_0 \\ x\neq y}} \frac{\|F'(x_0)^{-1}(F'(x) - F'(y))\|}{\|x - y\|} .$$

If $2L_0\|x_1 - x_0\| \leq 1$, then the Newton process (2.2) generates a sequence $\{x_n\} \subset \bar{S}_0$ which converges to the unique solution x^* of (2.1) in $\bar{S}_0$. If we put

$$L_n = \sup_{\substack{x,y\in\bar{S}_0 \\ x\neq y}} \frac{\|F'(x_n)^{-1}(F'(x) - F'(y))\|}{\|x - y\|}, \quad n \geq 1$$

and

$$d_n = \|x_{n+1} - x_n\|, \quad n \geq 0 ,$$

then the following error estimates hold:

$$\|x^* - x_n\| \leq \delta_n \equiv \frac{2d_n}{1 + \sqrt{1 - 2L_n d_n}} \quad (n \geq 0)$$

$$\leq \frac{L_n d_{n-1}^2}{1 + \sqrt{1 - (L_n d_{n-1})^2}} \quad (n \geq 1)$$

$$\leq \frac{L_{n-1} d_{n-1}^2}{1 - L_{n-1}d_{n-1} + \sqrt{1 - 2L_{n-1}d_{n-1}}} \quad (n \geq 1) \qquad (3.9)$$

$$\leq \frac{L_{n-1} d_{n-1}^2}{1 - L_{n-1}d_{n-1}} \qquad (n \geq 1) . \qquad (3.10)$$

<u>Proof</u>. Put $\varphi = 0$ in Theorem 3.1. Then Corollary 3.1.1 follows from Theorem 3.1 by noting that

$$d_n \leq \frac{1}{2} L_n d_{n-1}^2 \quad \text{and} \quad L_n \leq \frac{L_{n-1}}{1 - L_{n-1}d_{n-1}}, \quad n \geq 1. \qquad \text{Q.E.D.}$$

The bound (3.10) is due to Lancaster [8] and (3.9) is what Potra cited in his recent paper [17] as Kornstaedt's bound [7].

Corollary 3.1.2. Under the notation and assumptions of Corollary 3.1.1, we have

$$\|x^* - x_n\| \leq \delta_n \leq \frac{2d_n}{1 + \sqrt{1 - 2L_0(1 - L_0\Delta_n)^{-1}d_n}} \quad (n \geq 0)$$

$$\leq \frac{2\|F'(x_0)^{-1}F(x_n)\|}{1 - L_0\Delta_n + \sqrt{(1 - L_0\Delta_n)^2 - 2L_0\|F'(x_0)^{-1}F(x_n)\|}} \quad (n \geq 0) \tag{3.11}$$

$$\leq \frac{L_0 d_{n-1}^2}{1 - L_0\Delta_n + \sqrt{(1 - L_0\Delta_n)^2 - (L_0 d_{n-1})^2}} \quad (n \geq 1) \tag{3.12}$$

where $\Delta_n = \|x_n - x_0\|$.

Proof. It is easy to see that

$$L_n \leq \frac{L_0}{1 - L_0\Delta_n}$$

and

$$d_n \leq \|F'(x_n)^{-1}F'(x_0)\| \cdot \|F'(x_0)^{-1}F(x_n)\|$$

$$\leq \frac{1}{1 - L_0\Delta_n} \|F'(x_0)^{-1}F(x_n)\| .$$

Furthermore, we have

$$F'(x_0)^{-1}F(x_n) = F'(x_0)^{-1}\{F(x_n) - F(x_{n-1}) - F'(x_{n-1})(x_n - x_{n-1})\}$$

$$= F'(x_0)^{-1} \int_0^1 \{F'(x_{n-1} + t(x_n - x_{n-1})) - F'(x_{n-1})\}(x_n - x_{n-1})dt$$

so that

$$\|F'(x_0)^{-1}F(x_n)\| \leq \frac{1}{2} L_0 d_{n-1}^2 , \quad n \geq 1 .$$

Therefore, Corollary 3.1.2 follows from Corollary 3.1.1. Q.E.D.

Remark 3.1. The bounds (3.11) and (3.12) were recently obtained by Potra [17] under the assumptions of Theorem 2.1 as $\beta_5(\Delta_n, \|F'(x_0)^{-1}F(x_n)\|)$ and $\beta_4(\Delta_n, d_{n-1})$ respectively in his notation.

We can further improve the bound δ_n obtained in Corollary 3.1.1.

Corollary 3.1.3. Under the assumptions of Corollary 3.1.1, put $M_0 = L_0$ and for $n \geq 1$

$$\bar{S}_n = \bar{S}(x_n, 2\|F'(x_n)^{-1}F(x_n)\|) ,$$

$$M_n = \sup_{\substack{x,y \in \bar{S}_n \\ x \neq y}} \frac{\|F'(x_n)^{-1}(F'(x) - F'(y))\|}{\|x - y\|} .$$

Then we have

$$x^* \in \bar{S}_n \subseteq \bar{S}_{n-1} \subseteq \dots \subseteq \bar{S}_0 .$$

and

$$\|x^* - x_n\| \leq \varepsilon_n \equiv \frac{2d_n}{1 + \sqrt{1 - 2M_n d_n}} \leq \delta_n, \quad n \geq 0 .$$

Proof. This immediately follows from Lemma 2.6 by noting that $2d_{n+1} \leq d_n$ and

$$\|x^* - x_{n+1}\| \leq \frac{1}{2} M_n \|x^* - x_n\|^2 \leq \frac{1}{2} L_n \|x^* - x_n\|^2 , \quad n \geq 0 . \qquad \text{Q.E.D.}$$

Remark 3.2. As was remarked in [17], the cost of obtaining K_n, L_n or M_n might be very high. Therefore, in practical computation, it would be better to make use of one of (3.2), (3.3), (3.6) and (3.12). However, Theorem 3.1 and its Corollaries assert that the error bounds which have been obtained by many authors with the use of different techniques can be derived from the majorant theory of Kantorovich and Lemma 2.6, in a unified manner.

Theorem 3.2. Under the assumptions of Lemma 2.1, we have

$$d_n \leq \frac{\nabla t_{n+1}}{(\nabla t_n)^2} d_{n-1}^2 \leq \frac{\nabla t_{n+1}}{\nabla t_n} d_{n-1}$$

$$\leq \frac{1}{2 \cosh 2^{n-1}\varphi} d_{n-1} , \quad n \geq 1 . \tag{3.13}$$

The equality holds in (3.13) if and only if $\alpha h = 1$.

Proof. It follows from (2.6) and Lemma 2.3 (ii) that

$$d_n \leqq \frac{1}{2} KB_n d_{n-1}^2 = \frac{\nabla t_{n+1}}{(\nabla t_n)^2} d_{n-1}^2 \leqq \frac{\nabla t_{n+1}}{\nabla t_n} d_{n-1} . \tag{3.14}$$

Choose $\varphi^* \geqq 0$ such that $(1 + \cosh \varphi^*)h = 1$. Then, from Lemmas 2.4 and 2.5, we have

$$\nabla t_{n+1} = e^{2^n \varphi^*}(t^* - t_{n+1}) = \begin{cases} \dfrac{\sinh \varphi^*}{\sinh 2^n \varphi^*} \eta & (\varphi^* > 0) \\ 2^{-n}\eta & (\varphi^* = 0) . \end{cases}$$

Hence

$$\frac{\nabla t_{n+1}}{\nabla t_n} = \begin{cases} \dfrac{\sinh 2^{n-1}\varphi^*}{\sinh 2^n \varphi^*} & (\varphi^* > 0) \\ \dfrac{1}{2} & (\varphi^* = 0) \end{cases}$$

$$= \frac{1}{2 \cosh 2^{n-1}\varphi^*} \leqq \frac{1}{2 \cosh 2^{n-1}\varphi} \tag{3.15}$$

for every $\varphi \in [0, \varphi^*]$. The equality holds in (3.15) if and only if $\varphi = \varphi^*$. This, together with (3.14), proves Theorem 3.2. Q.E.D.

Remark 3.3. The bound (3.13) is of the form found in Ostrowski [16; Theorem 38.2].

4. Observation

Under the assumptions of Theorem 2.2, Ostrowski proved that

$$\frac{\|F(x_{n+m})\|}{\|F(x_n)\|} \leqq \begin{cases} \left(\dfrac{\sinh 2^{n-1}\varphi}{\sinh 2^{n+m-1}\varphi}\right)^2 & (\varphi > 0) \\ \dfrac{1}{2^{2m}} = \lim\limits_{\varphi \to +0} \left(\dfrac{\sinh 2^{n-1}\varphi}{\sinh 2^{n+m-1}\varphi}\right)^2 & (\varphi = 0) \end{cases} \tag{4.1}$$

$$\leqq \frac{1}{2^{2m}} , \quad n \geqq 0, \quad m \geqq 0 ,$$

provided that $F(x_n) \neq 0$. By our approach, we can easily derive his estimates. Let $\bar{K} = \sigma^{-1}$, $\bar{B} = \|F'(x_0)^{-1}\|$, $\bar{\eta} = \bar{B}\|F(x_0)\|$, $\bar{f}(t) = \frac{1}{2}\bar{K}\bar{B}t^2 - t + \bar{\eta}$. Furthermore, we define the sequences $\{\bar{B}_n\}$ and $\{\bar{\eta}_n\}$ by

$$\bar{B}_0 = \bar{B}, \quad \bar{\eta}_0 = \bar{\eta}, \quad \bar{h}_0 = \bar{K}\bar{B}_0\bar{\eta}_0,$$

$$\bar{B}_n = \frac{\bar{B}_{n-1}}{1 - \bar{h}_{n-1}}, \quad \bar{\eta}_n = \frac{\bar{h}_{n-1}\eta_{n-1}}{2(1 - \bar{h}_{n-1})}, \quad \bar{h}_n = \bar{K}\bar{B}_n\bar{\eta}_n, \quad n \geq 1 .$$

Then it is easy to see that

$$\|F'(x_n)^{-1}\| \leq \bar{B}_n , \quad \|x_{n+1} - x_n\| \leq \bar{\eta}_n$$

and

$$F(x_{n+1}) = F(x_n) + \int_0^1 F'(x_n + t(x_{n+1} - x_n))(x_{n+1} - x_n)dt$$

$$= \int_0^1 \{F'(x_n) - F'(x_n + t(x_{n+1} - x_n))\}F'(x_n)^{-1}F(x_n)dt .$$

Hence we have

$$\|F(x_{n+1})\| \leq \frac{1}{2}\bar{K}\|x_{n+1} - x_n\|\|F'(x_n)^{-1}\|\cdot\|F(x_n)\|$$

$$\leq \frac{1}{2}\bar{K}\bar{\eta}_n\bar{B}_n\|F(x_n)\| = \left(\frac{\bar{\eta}_n}{\bar{\eta}_{n-1}}\right)^2\|F(x_n)\| . \tag{4.2}$$

Define the sequence $\{\bar{t}_n\}$ by

$$\bar{t}_0 = 0, \quad \bar{t}_{n+1} = \bar{t}_n - \bar{f}(t_n)/\bar{f}'(t_n), \quad n \geq 0 .$$

Then, by Lemmas 2.2, 2.4 and 2.5, we have

$$\bar{\eta}_n = \bar{t}_{n+1} - \bar{t}_n = e^{2^n\varphi}(\bar{t}^* - \bar{t}_{n+1})$$

$$= \begin{cases} \dfrac{\sinh \varphi}{\sinh 2^n\varphi}\,\eta & (\varphi > 0) \\ 2^{-n}\eta & (\varphi = 0)\ . \end{cases} \tag{4.3}$$

since $\alpha\bar{h}_0 = 1$, where $\bar{t}^* = (1 - \sqrt{1 - 2\bar{h}_0})/\bar{K}$. Therefore, we have from (4.2) and (4.3)

$$\frac{\|F(x_{n+1})\|}{\|F(x_n)\|} \leq \begin{cases} \left(\dfrac{\sinh 2^{n-1}\varphi}{\sinh 2^n\varphi}\right)^2 & (\varphi > 0) \\ \dfrac{1}{2^2} & (\varphi = 0)\ . \end{cases}$$

This leads to the estimate (4.1). Therefore, together with (3.13) which holds in our case, we proved the main part of his theorem [16; Theorem 38.2]. The remaining part also follows from our approach.

Finally we remark that the chart for the lower bounds given in [28] is still true under the assumptions of Lemma 2.1 with a slight modification:

$$\|x^* - x_n\| \geqq \frac{2d_n}{1 + \sqrt{1 + 2K_n d_n}} \quad (n \geqq 0)$$

$$\geqq \begin{cases} \dfrac{2d_0}{1 + \sqrt{1 + 2h}} & (n = 0) \\ \dfrac{2d_n}{1 + \sqrt{1 + 2K\{1 - K(\|x_n - x_1\| + d_0)\}^{-1}d_n}} & (n \geqq 1) \end{cases} \tag{4.4}$$

$$\geqq \frac{2d_n}{1 + \sqrt{1 + 2K(1 - Kt_n)^{-1}d_n}} \quad (n \geqq 0) \quad \text{(Miel [11], Schmidt [23])}$$

$$\geq \frac{2d_n}{1 + \sqrt{1 + \dfrac{2Kd_n}{Kd_n + \sqrt{1 - 2h + (Kd_n)^2}}}} \quad (n \geq 0) \text{ (Potra-Ptak [18])}$$

$$\geq \frac{2d_n}{1 + \sqrt{1 + 2h_n}} \quad (n \geq 0) \quad \text{(Gragg-Tapia [3])}.$$

If the assumptions of Theorem 3.1 (iv) are satisfied, then (4.4) may be replaced by the sharper lower bound

$$\|x^* - x_n\| \geq \frac{2d_n}{1 + \sqrt{1 + 2K(1 - K\Delta_n)^{-1} d_n}} \quad (n \geq 0) \quad \text{(Yamamoto [28])} .$$

We also have

$$\|x^* - x_n\| \geq \varepsilon_n \equiv \frac{2d_n}{1 + \sqrt{1 + 2M_n d_n}} , \quad n \geq 0 ,$$

with the notation and assumptions of Corollary 3.1.3.

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On the basis of the results obtained in a series of papers [25] - [28], a convergence theorem for Newton's method in Banach spaces is given, which improves the theorems of Kantorovich [4], Lancaster [8] and Ostrowski [16]. The error bounds obtained improve the recent results of Potra [17].

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